Speeding up deferred acceptance*

Gregory Z. Gutin[†,‡] Daniel Karapetyan[§] Philip R. Neary[¶]
Alexander Vickery[‖] Anders Yeo[**] [††]

June 16, 2025

Abstract

A run of the deferred acceptance (DA) algorithm may contain proposals that are sure to be rejected. In this paper we introduce the *accelerated deferred acceptance* algorithm that proceeds in a similar manner to DA but with sure-to-be rejected proposals ruled out. Accelerated deferred acceptance outputs the same stable matching as DA but does so more efficiently: it terminates in weakly fewer rounds, requires weakly fewer proposals, and stable pairs match no later. Computational experiments show that the efficiency savings can be strict.

[*] We thank Sophie Bade, Claudia Cerrone, Arnaud Chevalier, Kim Sau Chung, Manolis Galenianos, Yannai Gonczarowski, Axel Gottfries, Michael Greinecker, Ori Heffetz, Bob Hammond, Áine Hyland, Onur Kesten, Alexey Kushnir, Melanie Lührmann, David Manlove, Myra Mohnen, Josué Ortega, Al Roth, Frances Ruane, Ran Shorrer, Joel Sobel, Alex Teytelboym, Ija Trapeznikova, Gregory Veramendi, Zaifu Yang, Gabriel Ziegler, seminar participants at the 2024 Irish Workshop on Economic Theory and RHUL Economics' lunchtime seminar, and, especially, Vince Crawford, for helpful comments.

[†] Computer Science Department, Royal Holloway University of London.
[‡] School of Mathematical Sciences and LPMC, Nankai University, Tianjin, China.
[§] School of Computer Science, University of Nottingham.
[¶] Economics Department, Royal Holloway University of London.
[‖] BDO LLP.
[**] IMADA, University of Southern Denmark.
[††] Department of Mathematics, University of Johannesburg.

1 Introduction

In this paper we introduce a new algorithm to find a stable matching in two-sided matching markets. Our *accelerated deferred acceptance* algorithm combines the classic deferred acceptance (DA) algorithm of Gale and Shapley (1962) with the iterated deletion of unattractive alternatives (IDUA) procedure (Balinski and Ratier, 1997; Gutin et al., 2023). Accelerated deferred acceptance is reminiscent of DA since it is based on sequential proposals, rejections, and tentative acceptances. Accelerated deferred acceptance borrows insight from IDUA by truncating preference lists so that future, sure-to-be rejected proposals are prevented from taking place.

We consider two-sided markets with "men" on one side and "women" on the other. With men in the role of proposers, accelerated deferred acceptance (hereafter ADA) diverges from classical DA in just one way: once a woman has a proposal, she rejects all men ranked below her top proposer and not just those that proposed concurrently.[1,2]

Algorithm Accelerated deferred acceptance

Initialise all men as single. Every round of the algorithm proceeds as follows:

1. Each single man proposes to his most preferred woman who has not yet rejected him.
2. Each woman with at least one proposal tentatively accepts her top proposer and rejects all **men** that she ranks below him.
3. Any man who is not matched becomes single.

When there is a round in which no man is rejected, return the current pairs.

Theorem 1 confirms that ADA always returns the same output as DA. Given this, any and all properties of DA are inherited by ADA "for free". For example, ADA generates the proposer-optimal stable matching and a mechanism based on it must be strategy proof for the proposers.

While both algorithms ultimately arrive at the same stable matching, the *pre-emptive rejections* allowed by ADA mean that it typically takes a different route to the shared endpoint. With ADA, once a woman is proposed

[1]Operationally, the difference is clear. Semantically, however, the algorithms are very similar. In fact, ADA deviates from DA (see Algorithm 1 for the formal statement) in just one term: the word "men", typeset in **red** for emphasis, replaces the word "proposers".

[2]It turns out that ADA is equivalent to the round-by-round variant of, and therefore a special case of, the *extended Gale-Shapley algorithm* of Gusfield and Irving (1989). The connection is detailed later in this section. We thank David Manlove for pointing this out.

to, she will never again receive a proposal from a man that she ranks below her current partner. As such, whenever a woman receives a new proposal, she is guaranteed to trade up. For men the situation is also different because with ADA a rejection may arrive out of the blue. Such pre-emptive rejections may be unexpected at the time, but, in the event that a man is single or will be at some point in the future, can be informative for deciding who he ought propose to next.

Not only is the alternate path taken by ADA to arrive at the man-optimal stable matching different to that of DA, it is also more efficient according to certain objective measures: number of proposals required for the market to clear; number of rounds for the market to clear; round in which equilibrium pairs first match. When benchmarked against DA's performance, ADA requires weakly fewer proposals (Theorem 2), takes weakly fewer rounds (Theorem 3), and equilibrium pairs match no later (a consequence of Theorem 4).

Theorems 2-4 guarantee no efficiency losses but are silent on the extent of the efficiency gains. We explore these via computational experiments on simulated data.[3] This is a non-trivial task because the number of two-sided matching markets of size n is $(n!)^{2n}$, and so even enumerating all of them is computationally infeasible for $n = 6$ and up. For this reason, we propose a novel market generator that allows one to sample random markets of any size. A scalar parameter, $c \in [0, 1]$, biases the sampling distribution. The value $c = 0$ corresponds to preferences drawn uniformly from the set of all preferences. At the other extreme, where $c = 1$, all individuals on the same side of the market have identical preferences, a property known as a "universal ranking" (Holzman and Samet, 2014).[4]

Our simulations show that ADA's efficiency savings can be substantial.[5] For example, in markets with 4,096 ($= 2^{12}$) participants on each side and $c = 0.9$, the average number of proposals used by DA is over 7,500,000 whereas, in contrast, ADA required only 208,000. This is a reduction of 96%. Similarly, in markets of size 1,024 ($= 2^{10}$) and $c = 0.9$, the average number of rounds needed by DA was 1,434 while ADA took only 91 before

[3]All simulations were run in Python. Code is publicly available at Karapetyan (2024).

[4]While statistically unlikely, a universal ranking is a common assumption in marriage markets with endowments (Cole et al., 1992; Burdett and Coles, 1997; Eeckhout, 1999). A universal ranking for one side is hardwired in to certain systems, of which an example is the centralised, third-level admissions system in the Republic of Ireland. Marks out of 625 in the state-administered "Leaving Cert" exam are the sole determinant of university entry because all programs are mandated to prioritise candidates in the same way: the higher a student's mark, the higher their priority.

[5]All results are averaged over 10,000 draws.

clearing. This reduction is 94%.

In a run DA and in a run of ADA, women only ever trade up. Given this, once a woman is matched she stays matched and, in particular, once a stable pair forms they remain together forever more. As regards our novel measure of the round in which stable pairs first match, again ADA fares noticeably better. Figure 1 plots this metric for both algorithms for a market with 1,024 participants and $c = 0.9$. ADA's trajectory is given by the solid line and DA's by the dashed one.

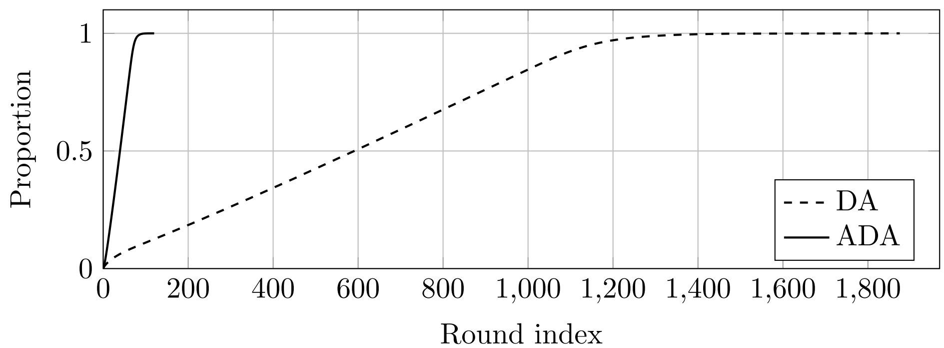

Figure 1: The proportion of final pairs matched by round.

By the time that ADA wraps up in round 123, DA is only 7% of the way to completion (it takes 1,820 rounds). Moreover, as of round 123, DA has only identified 13% of the equilibrium pairings; the remaining 87% of individuals are either unmatched or are currently paired with someone who they will not end up with. We note that the shape of both plots is concave: they increase linearly up until ∼95% stable pairings are found, followed by a tapering off. That is, each algorithm requires a (relatively) large number of rounds to find matches for the last ∼5% of individuals. In Section 4.4, we show that this shape is not an artefact of this class of preferences. Rather, it is common to every market that we considered. We have no good explanation for this feature nor do we know if it is universal, but it is striking.

We hope that the efficiencies of ADA over DA will be of general interest since variations of DA appear in labour markets (Crawford and Knoer, 1981; Kelso and Crawford, 1982), spectrum auctions (Milgrom and Segal, 2020), and school choice (Abdulkadiroğlu and Sönmez, 2003). Consider the savings from ruling out sure-to-be rejected proposals in each of these environments. An attempt to recruit a worker with a contract inferior to what they currently hold is unlikely to prove tempting. When participating in an auction, why bother submitting a bid below the current ask?[6] Filling out an application form for a school that is out of reach is nothing but a tax on the time of both parties. Whenever proposals are costly, then savings are possible. The reduction in proposals of ADA leads to a further reduction in the number of rounds. We refer to a round in which all proposals are rejected as an *idle round*. ADA never produces an idle round, whereas DA may include one and often contains many.

ADA's savings over DA may be magnified in markets more general than those considered here. As an example, consider an unbalanced market with n women and strictly more men. Consider a man who ends up unmatched (by definition there is at least one). A run of DA will last for at least n rounds because the above mentioned man proposes to each of the women on his preference list. But with ADA, this man may drop out in an earlier round as a result of pre-emptive rejections.

While ADA has some clear advantages, it should not be viewed as uniformly superior to DA. First of all, ADA's efficiency gains depend crucially on the notion of a "round".[7] If each operation within a round, i.e., a proposal or a rejection, has equal cost (a common algorithmic assumption), then ADA can have rounds that are more costly. Certainly, ADA requires (weakly) more operations in the first round: both have the same number of proposals, while ADA has at least as many rejections.

Second, a run of ADA uses weakly more total rejections (defined as the sum of direct rejections, made in response to a proposal, and pre-emptive rejections). Furthermore, we see no reason why the cost of communicating a direct rejection and that of communicating a pre-emptive one would always be equal. On the contrary, it seems plausible to us that these communication costs could vary with market structure. And depending on the relative costs, ADA may or may not be preferred to DA.

For the reasons above, we believe that ADA's advantages lie in settings that employ mechanisms that are dynamic, i.e., unmatched individuals are called upon to act every round. One such example is the *iterated deferred acceptance algorithm* of Bó and Hakimov (2022), a sequential mechanism, motivated by its real-world use in certain school assignment systems, wherein unmatched proposers make one proposal each round and pairings are made and updated according to proposals in each round.

We conclude the introduction with a brief discussion of interpretation. One can interpret ADA in a similar light to DA, as a setting where men

[6] In fact, often the rules of the exchange will explicitly forbid such behaviour.

[7] We thank an anonymous referee for urging greater emphasis of this point.

traverse a dance floor to ask women for a dance, but with the algorithm clever enough that it can identify wasteful proposals coming down the tracks and barring them. Other readings are possible. One way to make sense of the difference in the two procedures is in the level of informational feedback.[8] Under DA, men only receive feedback in response to a rejected proposal and any time that a man receives feedback it is of the form: "your proposee rejected you". The information flow associated with ADA is richer in two ways. First, two kinds of feedback for men are possible: "your proposee rejected you" and "the following women are out of your league". Second, the timing of feedback is less structured. While the former occurs only in response to a rejected proposal, the latter can arrive in any round and is irrespective of the whether the man is matched or unmatched.

In order for an increase in information feedback to improve efficiency, individuals must act on it. But this requires some level of strategic sophistication. Gale and Shapley's DA demands very little of market participants: men blindly propose to their top available choice and all that is required of women is that they can rank proposers.[9]

In contrast, our ADA can be understood as participants displaying sophisticated reasoning. In the first round each man proposes to his ideal partner; this seems reasonable since by definition she is unattached. But in later rounds, mightn't a strategic proposer contemplate making a proposal to a woman who already has a partner? And if proposals are costly to make, some reflection on the part of would-be proposers could well rule out sure-to-be rejected ones. Along these lines, Bó and Hakimov (2019) report on an experimental test of the iterated deferred acceptance algorithm with subjects who, from round 2 onwards, avoid proposals that won't lead anywhere; they refer to this behaviour as a "skipping strategy". Furthermore, just as making a proposal might be costly, it is possible that responding to one is burdensome. If this is the case, then it is in a woman's interest to avoid unnecessary proposals. It seems reasonable to suppose that protocols or mechanisms might develop whereby women convey this information to those who are certain to waste their time in future.

[8] The role that information plays in markets cannot be understated, Hayek (1945) is the classic reference, and matching markets are no exception. Information feedback seems particularly relevant for dynamic environments (Akbarpour et al., 2020; Doval, 2022).

[9] Proposing to one's top available choice without any thought has the flavour of a greedy algorithm, and in the language of models of strategic sophistication (Nagel, 1995; Stahl and Wilson, 1994, 1995) might best be described as level-0 behaviour.

Related literature

It is difficult to overstate the role played by deferred acceptance in the development of market design. In fact, an argument can be made that the two are inseparable. Roth (2008) is a survey article devoted entirely to the algorithm that documents its importance and provides historical context. The 2023 edited volume *Online and Matching-Based Market Design* (Echenique et al.) contains 32 chapters written by leading experts. Discussion of DA appears throughout and plays a prominent role in Chapters 1, 7, 8, 10, 11, 12, 14, 15, and 24. Direct reference to "deferred acceptance" or "DA" occurs in excess of 300 times.

While we view our augmentation of Gale and Shapley's original 1962 DA algorithm as the main contribution of this paper, it is important to note that we are not as original as we first thought. Indeed, it turns out that accelerated deferred acceptance algorithm is simply the round-by-round variant of the *extended Gale-Shapley algorithm* proposed in Gusfield and Irving (1989). The difference can be understood as follows. Like Gale and Shapley's original formulation of DA, our ADA proceeds via rounds wherein all single men simultaneously propose. Gusfield and Irving's procedure, on the other hand, uses the *nondeterministic* proposal protocol of the formulation of DA due to McVitie and Wilson (1971), in which only one unmatched man makes a proposal in any given step.[10]

Every run of ADA has an equivalent run of Gusfield and Irving's procedure since some nondeterministic sequence of proposals can be found that matches up with ADA's ordering. Moreover, what we believe is the key insight of ADA — not allowing sure-to-be rejected proposals — clearly belongs to Gusfield and Irving. That being said, we believe that ADA remains worthy of study. First, procedures based on simultaneous operations are a closer fit to the higher ordering reasoning systems that economists and game theorists assume of rational agents. Second, a simultaneous proposal protocol is the implementation of choice for many assignment programs (see Bó and Hakimov (2019, 2022) for examples in school assignment).

The efficiency savings of ADA over DA will be magnified by substituting ADA into mechanisms that require running DA more than once. One such procedure is the *efficiency-adjusted deferred acceptance* (EADA), due

[10]Knuth (1996) notes that McVitie and Wilson's implementation of DA is closely related to the *coupon collector problem* that is described as follows: if each box of your favourite cereal contains exactly one of n distinct coupons, then how many boxes of cereal must you buy, on average, before you are in possession of all n coupons? Knuth uses this connection to compute the mean and variance of proposals required for the market to clear.

to Kesten (2010), that was introduced to counteract unnecessary welfare losses that appear with DA (in particular when applied to school choice).[11] EADA revises allocations whenever they give rise to a rejection cycle, an operation that Tang and Yu (2014) show is equivalent to iteratively running DA on appropriately defined submarkets.

Gonczarowski et al. (2023) are concerned with the policy-relevant issue of how best to demonstrate DA's strategy proof-ness without sacrificing simplicity. They observe that every strategy proof mechanism has a *menu description* (Hammond, 1979) and they argue that DA's strategy proof-ness is more easily conveyed using this approach.[12] This is relevant because the procedure that generates menus involves multiple runs of DA.

Another prominent and well-studied mechanism for assigning students to schools is the *Boston mechanism*. This involves collecting reported preferences and assigning students to schools over multiple rounds. The chief difference between the Boston mechanism and DA is that with Boston acceptances are not tentative but final: once a student is assigned to a school, they are in. This feature leaves open the possibility that students might try to game the system by misreporting preferences, because a student who is rejected by their top choice school in round 1 might find that their second choice is no longer available in round 2. Mennle and Seuken (2017) describe the *adaptive Boston mechanism*, used in certain parts of Germany, that proceeds in rounds but, unlike the Boston mechanism, with feedback: in every round, unmatched students are informed of the schools that still have seats available. Like ADA, this rules out sure-to-be rejected proposals.

Structure of the paper

The balance of the paper is as follows. Section 2 fixes notation and reminds the reader of the environment, the DA algorithm, and the IDUA procedure. In Section 3 we introduce the accelerated deferred acceptance algorithm and present its theoretical properties. Section 4 describes our market generator and presents the results of our simulations. Section 5 concludes and discusses potential avenues for future work.

[11] Cerrone et al. (2024) report on an experimental horserace between EADA and DA. They find that efficiency is improved and truth-telling rates are higher under EADA even though it is not strategy proof. Despite its guaranteed efficiency advantages, EADA only improves outcomes for students who are already "doing ok" (Ortega et al., 2024). Specifically, Ortega et al. show that EADA does not improve the fortunes of students who are assigned to their worst-ranked schools or those who remain unmatched under DA.

[12] Gonczarowski et al. (2024) describe an experiment that corroborates this conjecture.

2 Preliminaries

In Section 2.1 we define the environment under study. (It is precisely the one-to-one two-sided matching framework introduced by Gale and Shapley.) In Section 2.2 we recap the original *deferred acceptance* (DA) algorithm and the *iterated deletion of unattractive alternatives* (IDUA) procedure of Balinski and Ratier (1997) and Gutin et al. (2023).

2.1 Matching problems

Let M be a set of n men and let W be a set of n women, where n is a positive integer greater than or equal to 2. Each man $m \in M$ has a strict preference relation, $\succ_m$, over the set of all women, and each woman $w \in W$ has a strict preference relation, $\succ_w$, over the set of all men. When man m prefers woman w' to woman w'', we write $w' \succ_m w''$, with an analogous statement for the preferences of women. (Occasionally it will be more convenient to present preferences as linearly ordered lists where the first entry on the list is the most preferred, and so on. Which notation is being employed will be clear from context.)

Definition 1. A matching problem, P, is a tuple $(\{\succ_m\}_{m\in M}, \{\succ_w\}_{w\in W})$, where $\{\succ_m\}_{m\in M}$ and $\{\succ_w\}_{w\in W}$ are the collections of preferences, one for each of the n men and one for each of the n women.

A *matching* in P is a mapping μ from $M \cup W$ to itself such that: for every man $m \in M$, $\mu(m) \in W$; for every woman $w \in W$, $\mu(w) \in M$; and for every man-woman pair $(m, w) \in M \times W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

The following is the key definition proposed by Gale and Shapley.

Definition 2. Consider a matching μ in a matching problem P. Man m and woman w form a *blocking pair* with respect to μ if $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

In words, a man-woman pair (m, w) is blocking with respect to matching μ if both m and w prefer each other over their partners in μ. Gale and Shapley defined stability by the absence of a blocking pair.

Definition 3. A matching μ in P with no blocking pairs is a *stable matching*.

Gale and Shapley then proved the following remarkable result.

Theorem (Gale and Shapley (1962))**.** Every stable matching problem possesses at least one stable matching.

2.2 DA and IDUA

In this section we recap the DA algorithm and the IDUA procedure. We document both since ADA is a blend of the two.

Before beginning, one key difference between the two procedures is worth highlighting. The DA algorithm returns a stable matching. The IDUA procedure, on the other hand, generates the *normal form*: a (weakly) smaller matching market with the same set of stable matchings as the original that is, in a sense, "boxed in" by the two extreme stable matchings. IDUA only generates a stable matching in the event that the original market possesses exactly one in which case the normal form is the unique stable matching and nothing more (see the main result of Gutin et al. (2023)).

2.2.1 Deferred acceptance

In a run of DA, one side of the market makes proposals that the other side reacts to. We frame it such that men propose and women respond.

DA begins by initialising all men as single. Each round of the algorithm has three steps. First, each single man proposes to the woman currently top of his preference list. Second, each woman that received at least one proposal tentatively accepts the top proposer and rejects the other proposers. Third, each man who was rejected udpates his preference list by removing from it the woman that he proposed to and is deemed single for the next round.

Algorithm 1 DA algorithm

Initialise all men as single. Each round of the algorithm proceeds as follows:

1. Each single man proposes to his most preferred woman who has not yet rejected him.
2. Each woman with at least one proposal tentatively accepts her top proposer and rejects all **proposers** that she ranks below him.
3. Any man who is not matched becomes single.

When there is a round in which no man is rejected, return the current pairs.

That a stable matching exists for every two-sided matching market is a consequence of the following result.

Theorem (Gale and Shapley (1962))**.** The output of the man-proposing deferred acceptance algorithm is a stable matching.

2.2.2 Iterated deletion of unattractive alternatives

Gale and Shapley deemed a matching unstable if there exists a blocking pair with respect to it. According to this view, if a man and a woman who are matched but not matched together fare better outside the matching, then the matching's structural integrity is under threat. Gutin et al. (2023, 2024) put forward an alternative interpretation of instability. They argue that the causes of instability can be attributed to issues with pairs that are contained within a matching, specifically the two pairs that contribute to the breakaway couple. These pairs can be thought of as "weak" with respect to the matching.

A key point is that some pairs are *always* weak no matter the rest of the matching. To see how, suppose that woman w is man m's dream partner. Then there cannot exist a stable matching in which w is paired with a man that she ranks below m. Why? Because w could always (at least) propose to breakaway with m and m would certainly accept. If stability is a requirement, such pairs can be deleted as they are irrelevant. That is, men that w ranks below m can be removed from w's preference list and for each of these men w can be removed from his preference list.

The above motivates the notion of *unattractive alternatives.*

Definition 4 (Unattractive alternatives)**.** We say that man m is an unattractive alternative to woman w, if there is some man $m' \neq m$ such that (i) $m' \succ_w m$, and (ii) $w \succ_{m'} w'$ for all $w' \neq w$. A mirror-image statement describes when a woman is an unattractive alternative to a man. Finally, we say that w is an unattractive alternative to man m whenever m is an unattractive alternative to woman w (and vice versa).

Deleting unattractive alternatives leaves behind a smaller matching market that, almost by definition, has the same set of stable matchings as the original. More importantly, the deletion operation can iterate: suppose that woman w is the top choice for two men, m_1 and m_2, and suppose further that $m_1 \succ_w m_2$. While w's preference list will be truncated from m_1 on down, note that m_2's preference list will be truncated from the top since w is removed. Since a match with w won't happen, m_2 has a new most-preferred woman, say w', on his updated list. This may provide a new outside option for w' which might in turn allow w' to truncate her preference lists in a way that she was unable to before she became m_2's top choice.

The above describes the beginnings of the *iterated deletion of unattractive alternatives* (IDUA). This procedure repeatedly prunes redundant information from the preference lists as described above. IDUA continually deletes unattractive alternatives from preference lists until there remains no pair who view each other as unattractive. The procedure stops when no further deletions are possible and the (sub)matching market that remains is referred to as the *normal form*.[13]

Definition 5 (The iterated deletion of unattractive alternatives (IDUA))**.** Given a matching problem, $P = \left(\{\succ_m\}_{m\in M}, \{\succ_w\}_{w\in W} \right)$, we define $\succ_m^0 := \succ_m$ and $\succ_w^0 := \succ_w$, and for each $k \geq 1$, form the matching (sub)problem $P^k = \left(\{\succ_m^k\}_{m\in M}, \{\succ_w^k\}_{w\in W} \right)$ where for every man m and every woman w,

$$\begin{aligned} \succ_m^k &= \left\{ w \mid w \in \left\{ \succ_m^{k-1} \right\} \text{ and } m \in \left\{ \succ_w^{k-1} \right\} \right\}, \text{ and} \\ \succ_w^k &= \left\{ m \mid m \in \left\{ \succ_w^{k-1} \right\} \text{ and } w \in \left\{ \succ_m^{k-1} \right\} \right\}. \end{aligned} \tag{1}$$

Finally, define the *normal form* of matching problem P, denoted P^*, as P^{k^*} where k^* is the minimum k such that $P^{k+1} = P^k$. Men's preferences on the normal form are denoted $\left\{ \succ_m^* \right\}$ and those of women by $\{\succ_w^*\}$.

The following lemma confirms that, for an analyst interested in the set of stable matchings, restricting attention to the normal form is sufficient.

Lemma 1 (Balinski and Ratier (1997); Gutin et al. (2023))**.** The iterated deletion of unattractive alternatives does not change the set of stable matchings. That is, P and its associated normal form, P^*, contain exactly the same set of stable matchings.

The next lemma shows that the normal form is "boxed in" by the man-optimal and woman-optimal stable matchings. Before stating the lemma, we introduce some notation. Given a matching problem, P, for each man m, let $\tau(m)$ denote the woman at the top of m's truncated preference list in the normal form, P^*, and similarly, for each w, let $\tau(w)$ denote the man at the top of w's truncated list.

Lemma 2 (Balinski and Ratier (1997); Gutin et al. (2023))**.** Let P be stable matching problem and let P^* denote its normal form. The following two collections of pairs, μ_M and μ_W, are stable matchings in P.

$$\begin{aligned} \mu_M &= \left\{ \left(m_1, \tau(m_1)\right), \ldots, \left(m_n, \tau(m_n)\right) \right\} \\ \mu_W &= \left\{ \left(\tau(w_1), w_1\right), \ldots, \left(\tau(w_n), w_n\right) \right\} \end{aligned}$$

[13] Section 2.3 of Gutin et al. (2023) argues that IDUA is the matching market analog of the *iterated deletion of dominated strategies* procedure for strategic games. The reason for the parallel is that both leave behind a smaller mathematical object without changing the set of "solutions".

Consider the collection of pairs μ_M. To see that μ_M is a matching, observe that every man must have a different most preferred woman in the normal form. If not, then two men have the same most-preferred woman. But this cannot be because a woman is never indifferent over two men and so would cut loose the less preferred one, meaning that the IDUA procedure is not yet finished. To see that the matching μ_M is stable, note that every man is paired with his most preferred feasible woman (any preferred woman is infeasible because she does not appear on his reduced preference list). Such a matching must be stable because no man is willing to swap and so there can be no blocking pairs.

A mirror image argument to the above confirms that μ_W is a stable matching. The matchings μ_M and μ_W are typically referred to as the *man-optimal stable matching* and the *woman-optimal stable matching* respectively.

We conclude this section with two further comments on the normal form. First, the environment that we consider in this paper is a balanced one-to-one matching market where all preference lists are complete. Given this, all participants appear in some stable matching and hence appear in the normal form. But upon reflection, the normal form is always a balanced market regardless of the specifics of the original market and so, at least when it comes to the analysis of stable outcomes, restricting attention to balanced matching markets is not such a limitation after all.[14]

Second, Gutin et al. (2024) show that the same IDUA procedure that generates the normal form can also be used to explore the set of stable matchings in it. They consider *assignment constraints* and show how IDUA can be used to determine whether a stable matching satisfying the constraints is feasible and if so will output all of them.[15] In the absence of any constraints, the algorithm of Gutin et al. returns every stable matching.

3 Accelerated deferred acceptance

In this section we introduce the *accelerated deferred acceptance* algorithm. ADA is strikingly similar to the DA algorithm of Gale and Shapley. The key difference is that ADA moves at greater pace through the market because

[14]That the normal form is always a balanced market strengthens the *rural hospitals* theorem (McVitie and Wilson, 1970; Roth, 1984, 1986; Gale and Sotomayor, 1985a,b): not only does the normal form identify those participants that appear in every stable matching, it also identifies pairs that are not possible amongst these participants.

[15]Assignment constraints generalise the notion of *forced pairs* (Gusfield and Irving, 1989) and *restricted pairs* (Dias et al., 2003). That is, pairs that must / must not be included.

preferences are truncated much as they are in IDUA. Accelerated deferred acceptance can in fact be interpreted as a one-sided variant of IDUA.

Section 3.1 defines accelerated deferred acceptance and confirms that it generates the same outcome as DA. (Given this, accelerated deferred acceptance inherits all properties of classical DA, e.g., it is strategy proof for the proposing side.) In Section 3.2, we show that accelerated deferred acceptance has practical improvements over DA: it requires fewer proposals (Theorem 2), it terminates in fewer rounds (Theorem 3), and all tentative matches take place no later (Theorem 4). These features are illustrated by Example 1, that details a run of both algorithms on the same market.

3.1 The algorithm

The accelerated deferred acceptance algorithm is similar to DA except that whenever a woman receives a new a proposal, she rejects all men who she ranks below the most preferred proposing man, not just those who have also proposed to her.[16] Some of these rejections may be pre-emptive in that a subset of men are forbidden from ever proposing to certain women. Some of these men might have ultimately proposed while others might have never got around to it; but all are prevented from doing so.

Algorithm 2 Accelerated deferred acceptance algorithm

Initialise all men as single. Every round of the algorithm proceeds as follows:

1. Each single man proposes to his most preferred woman who has not yet rejected him.
2. Each woman with at least one proposal tentatively accepts her top proposer and rejects all **men** that she ranks below him.
3. Any man who is not matched becomes single.

When there is a round in which no man is rejected, return the current pairs.

Our main result in this section, Theorem 1, confirms that accelerated deferred acceptance finds the same stable matching as that found by DA. The following two observations, that also hold for a run of DA, are used to prove the result.

Observation 1. Men propose to women in decreasing order of preference.

[16] We remind the reader that the key difference between Gale and Shapley's DA algorithm (Algorithm 1) and our accelerated deferred acceptance (Algorithm 2) is one word: accelerated deferred acceptance replaces the word "proposers" in DA with the word "men". For emphasis we have typeset the updated word in **red**.

Observation 2. Once a woman is matched, she never becomes unmatched.

Observation 1 is a restatement of the fact that each man works down his preference list under ADA. However, with ADA a man may not work incrementally down his list as with DA, because it may be that the woman who would be top of his "as yet unproposed to" list has already pre-emptively rejected him. By Observation 2, women only ever "trade up". Again, while the same statement holds for DA, there is a difference. For ADA, whenever a woman receives a new proposal in a given round she is guaranteed of trading up because proposals only ever arrive from men who she prefers to her current match.

We now state the result.

Theorem 1. The man-proposing accelerated deferred acceptance returns the same matching as the man-proposing deferred acceptance algorithm.

We refer the reader to Gusfield and Irving (1989, §1.2.4) for a proof of Theorem 1. Gusfield and Irving show that the result holds for any sequence of proposals, a property they refer to as *nondeterminism*, and so clearly it holds for ADA since its round-by-round nature is only consistent with certain orderings of proposals.

Given that ADA and DA always return the same stable matching, any and all properties of DA, be they desirable or not, are assured to hold for ADA. For example, accelerated deferred acceptance must return the proposer-optimal stable matching. Moreover, since DA is *strategy proof* for those proposing (Dubins and Freedman, 1981; Roth, 1982), accelerated deferred acceptance is too.[17]

3.2 Fewer proposals, fewer rounds, and earlier pairings

In this section we compare and contrast both algorithms along various criteria. We begin with an example to illustrate how ADA can "move more quickly" through a matching market than DA.

Example 1. Consider a market with five men, $M = \{m_1, m_2, m_3, m_4, m_5\}$, and five women, $W = \{w_1, w_2, w_3, w_4, w_5\}$. The preference list for each man

[17] In the context of two-sided matching markets, an *allocation rule* is a mapping from the reported preferences of all $2n$ market participants to the set of matchings. An allocation rule is said to be *strategy proof* if it is a weakly dominant strategy for participants to report their preferences truthfully. For a concise proof that DA is strategy proof for the proposing side see Nisan et al. (2007, §10.4, page 258).

and each woman are presented below, with the participants on the other side of the market listed in order of decreasing preference.

$$\begin{array}{llll} w_1 : m_5, m_4, m_1, m_2, m_3 & \quad & m_1 : w_1, w_2, w_3, w_4, w_5 \\ w_2 : m_1, m_3, m_2, m_4, m_5 & & m_2 : w_1, w_4, w_5, w_2, w_3 \\ w_3 : m_5, m_4, m_3, m_2, m_1 & & m_3 : w_1, w_4, w_3, w_5, w_2 \\ w_4 : m_4, m_2, m_1, m_3, m_5 & & m_4 : w_4, w_2, w_3, w_1, w_5 \\ w_5 : m_5, m_1, m_3, m_4, m_2 & & m_5 : w_5, w_4, w_1, w_2, w_3 \end{array} \tag{2}$$

We now show how both DA and accelerated deferred acceptance operate on the market above. In the interest of space we will only present how both algorithms operate in the first two rounds. In fact, two rounds is enough for accelerated deferred acceptance to terminate but not so for DA as it requires four rounds. How DA continues to completion is spelled out in Appendix B.

Both DA and accelerated deferred acceptance begin with initialising all men as single. To indicate this in (2), we have written men in blue if they are active in ADA and circled a man in a blue circle if he is active in DA. All men are active in Round 1 of both so each man m_i is written as m_i.

In the first round each man approaches his top ranked woman. This is depicted in the Table 1 below. Below each woman are two columns that convey whether the woman is proposed to under DA or ADA. The column in gray is for DA and the column in green is for ADA. Whenever a woman receives multiple proposals we order the suitors in the column from top to bottom in order of decreasing preference. In this example, woman w_1 received multiple proposals in Round 1. She will tentatively accept her most preferred of these, man m_1. All tentative acceptances are represented by labelling the accepted man with an asterix.

Table 1: proposals and tentative acceptances in Round 1.

w_1		w_2		w_3		w_4		w_5	
m_1^*	m_1^*					m_4^*	m_4^*	m_5^*	m_5^*
m_2	m_2								
m_3	m_3								

Table 1 listed proposals and acceptances in Round 1. Now let us turn to rejections, which is the first point at which the two algorithms diverge. We represent rejections by updating the preference lists as in (3). We denote rejections that occur under ADA in red. Similarly, a rejection that occurs under DA is depicted by putting a red box around the relevant party. For example, since man m_2 was rejected by woman w_1 in both algorithms, we write $\boxed{m_2}$ in w_1's preference list and write $\boxed{w_1}$ in m_2's preference list.

Note the preference lists of w_4 and w_5 in (3) . Each woman was proposed to by her top ranked male, m_4 for w_4 and m_5 for w_5. Both algorithms therefore include the two tentative pairs (m_4, w_4) and (m_5, w_5). But for ADA there are pre-emptive rejections handed out by w_4 and w_5. It is for this reason that some entries on their respective preference lists are in red.

$$
\begin{array}{ll}
w_1 : m_5, m_4, m_1, \boxed{m_2}, \boxed{m_3} & m_1 : w_1, w_2, w_3, w_4, w_5 \\
w_2 : m_1, m_3, m_2, m_4, m_5 & m_2 : \boxed{w_1}, w_4, w_5, w_2, w_3 \\
w_3 : m_5, m_4, m_3, m_2, m_1 & m_3 : \boxed{w_1}, w_4, w_3, w_5, w_2 \\
w_4 : m_4, m_2, m_1, m_3, m_5 & m_4 : w_4, w_2, w_3, w_1, w_5 \\
w_5 : m_5, m_1, m_3, m_4, m_2 & m_5 : w_5, w_4, w_1, w_2, w_3
\end{array}
\tag{3}
$$

Some reflection reveals that ADA always starts out with weakly more rejections than DA. This is the case because any man rejected in Round 1 of DA is also rejected in Round 1 of ADA, but ADA also allows pre-emptive rejections to be handed out.

Let us now move to Round 2. The only single men at the beginning of this round are m_2 and m_3. DA stipulates that both men propose to the top ranked as-yet-unproposed-to woman on their list. From (3), we can see that for both men this is w_4. For ADA there is a difference. Consider m_2. During Round 1, he received a pre-emptive rejection from w_4 and w_5, conveyed by writing w_4 and w_5 in the preference list of m_2 in (3). Therefore m_2 proposes to w_2 in Round 2.

The full list of proposals in Round 2 are as in Table 2 below.

Table 2: proposals and tentative acceptances in Round 2.

w_1		w_2		w_3		w_4		w_5	
m_1^*	m_1^*		m_2^*		m_3^*	m_4^*	m_4^*	m_5^*	m_5^*
						m_3			
						m_2			

Note from Table 2 above that ADA is finished at the end of Round 2.

It has generated a collection of five tentative pairs $\{(m_i, w_i)\}_{i=1}^{5}$ that make up a matching, and Theorem 1 assures that whenever the algorithm finds a matching that the matching is stable.

In contrast to ADA, the DA algorithm does not terminate after two rounds. In Round 2 of DA, both m_2 and m_3 propose to woman w_4. But w_4 is already tentatively paired with m_4 who she prefers over each. So w_4 rejects these proposals and both m_2 and m_3 are back on the market in Round 3 with their preference lists updated as in (4). After 4 rounds, DA ultimately terminates at the same stable matching as that found by ADA. The details are provided in Appendix B.

$$
\begin{array}{ll}
w_1 : m_5, m_4, m_1, \boxed{m_2}, \boxed{m_3} & m_1 : w_1, w_2, w_3, w_4, w_5 \\
w_2 : m_1, m_3, m_2, m_4, m_5 & m_2 : \boxed{w_1}, \boxed{w_4}, w_5, w_2, w_3 \\
w_3 : m_5, m_4, m_3, m_2, m_1 & m_3 : \boxed{w_1}, \boxed{w_4}, w_3, w_5, w_2 \\
w_4 : m_4, \boxed{m_2}, m_1, \boxed{m_3}, m_5 & m_4 : w_4, w_2, w_3, w_1, w_5 \\
w_5 : m_5, m_1, m_3, m_4, m_2 & m_5 : w_5, w_4, w_1, w_2, w_3
\end{array} \tag{4}
$$

Let us now make some observations about the example.

After two rounds the number of proposals made by DA and ADA must be the same since each generates n proposals in the first round, and all males rejected in the first round make a new proposal in the second. This holds for our example with both m_2 and m_3 single at the beginning of Round 2.

Note that in Round 2 of the DA algorithm, all proposals made were rejected. We refer to a round of this kind as an *idle round*, since the collection of matched pairs did not update. The ADA algorithm cannot have an idle round. In Section 4 we show that DA can have many.

Woman w rejecting a proposal from man m can be interpreted as a deletion of m from the preference list of w and a removal of w from that of m. Viewed in this way, the structure of the preference lists upon termination of both algorithms is worth noting. DA leaves men's preferences as a subset of their original list that is contiguous from their stable match down to their least favourite woman; women's remaining preferences are a subset of the original list that need not be be contiguous. ADA is the opposite: the remaining preference list for a woman is a contiguous subset of the original list starting with the most preferred man and concluding with the stable match; men's remaining preference lists can be scattered.

Let us now consider our measures of efficiency. We begin with the number of proposals needed for each algorithm to terminate. From the analysis above we see that ADA required seven proposals to identify the man-optimal stable matching: five proposals in Round 1 and two proposals in Round 2. From Appendix B, we see that DA required ten proposals. It turns out that ADA requiring fewer proposals than DA is not a function of the specifics of Example 1. Rather, this is a general property that always holds, as confirmed by the following result that is proved in Appendix A.

Theorem 2. Accelerated deferred acceptance never requires more proposals than DA.

Our next efficiency measure is number of rounds required to completion. In Example 1, ADA needed two rounds whereas DA required four rounds, one of which was an idle round. Again, this efficiency gain is not an artefact of Example 1. Rather, we have the following assurance.

Theorem 3. Accelerated deferred acceptance always terminates in weakly fewer rounds than DA.

In fact, Theorem 3 above follows almost immediately from the following, stronger result.

Theorem 4. Each proposal that is made when running ADA takes place in the same or an earlier round than for DA.

It is clear that ADA is weakly "ahead" of DA after only one round of each algorithm since both contain the same number of direct rejections and ADA may contain some pre-emptive rejections too. Theorem 4 confirms that ADA stays weakly ahead. The proof of Theorem 4 is in Appendix A.

Theorem 3 follows from Theorem 4 because all final pairs match no later for ADA. That is, ADA terminates when the last final pair is matched and this happens on the back of an accepted proposal.

We conclude this section by mentioning the recent paper of Gokhale et al. (2024) that focuses on markets that generate worst case performance of classical DA as measured both by number of rounds and by number of proposals. The authors show that if a market maximises the number of rounds needed for DA to terminate then it also maximises the number of proposals required. Moreover, any such a market must have a unique stable matching. Similar questions regarding ADA can also be posed but we leave this to future work.

4 Computational experiments

Our theoretical findings in Section 3 confirm that ADA is weakly more efficient than DA in certain ways. However, the results are completely silent on the extent, if any, of the efficiency gains. The computational experiments described in this section not only confirm that improvements are possible, but that they can be sizeable.

Specifically, with preference lists pre-defined, the objectives of our computational study are as follows:

- study the number of rounds and the number of proposals generated by ADA compared to DA.
- study the proportion of stable pairs by round of each algorithm, which is a new metric that captures when each stable pair first matched.
- compare the running times of the two algorithms.

Section 4.1 defines the pseudo-random market generator and Sections 4.2, 4.3, 4.4 and 4.5 present our findings.

4.1 Market generator

Since the number of two-sided matching problems of size n is $(n!)^{2n}$, an exhaustive exploration of how the algorithms perform on every market is intractable from $n = 6$ and up. For this reason, we propose a novel market generator that allows one to sample random markets for any market size n where the bias in the sampling is controlled by a parameter $c \in [0, 1]$.

Let a market be described by a tuple $(\boldsymbol{W}, \boldsymbol{M})$, where $\boldsymbol{W} = (w_{i,j})$ and $\boldsymbol{M} = (m_{i,j})$ are $n \times n$ matrices. Each row of each of these matrices is a permutation of $(1, 2, \ldots, n)$.

Definition 6 (Market generator)**.** Our market generator takes two parameters: a non-negative integer n for the size of the market and a coefficient $c \in [0, 1]$ that controls the bias in the sampling distribution over preferences of participants on the same side of the market.[18] The generator works as follows.

1. Create a random permutation $\boldsymbol{p}$ of values $(0, 1, \ldots, n-1)$.
2. For each $i \in \{1, 2, \ldots, n\}$, create a random vector $\boldsymbol{v}_i \in \mathbb{R}^n$, where each element of $\boldsymbol{v}_i$ is in the range $[0, n-1]$.

[18]We assume that the bias is the same for both sides. This can be relaxed.

3. Calculate $\boldsymbol{u}_i = (1-c)\boldsymbol{v}_i + c\boldsymbol{p}$. Let $(m_{i,1}, m_{i,2}, \ldots, m_{i,n})$ be the permutation that orders $\boldsymbol{u}_i$ in ascending order.

4. Follow steps 1–3 to produce matrix $\boldsymbol{W}$.

Observe that the matrices $\boldsymbol{M}$ and $\boldsymbol{W}$ are uniformly random when $c = 0$. This follows because each row is drawn uniformly from the set of all possible preferences and the rows are pairwise independent. Conversely, when $c = 1$, the preferences of all the men are identical and the preferences of all the women are identical (Holzman and Samet refer to such preferences as those with a "universal ranking").

The implementation of our market generator is publicly available at Karapetyan (2024).

4.2 Number of rounds

The number of rounds and the number of proposals are arguably the most important metrics as it is easy to imagine a market where there are costs associated with each round and each proposal.

Figure 2 shows how the number of rounds changes with the the size of the market for three different values of the bias parameter: *uniformly random* ($c = 0$), *moderately similar* ($c = 0.5$), or *similar* ($c = 0.9$). The values of n change as $n = 2, 4, 8, \ldots, 4,096$ (i.e., ranging from 2^1 to 2^{12}). The lines are colour-coded according to the value of c. For each colour (value of c), the dashed lines show the average number of rounds taken under DA, while the solid lines plot the corresponding value for ADA.

Note that, unless specified otherwise, each point in every figure in this section corresponds to a mean value over 1,000 markets generated with identical parameter values but distinct pseudo-random number generator seeds.

For all three values of c, the total number of rounds for DA increases super-linearly with market size. In contrast, the number of rounds always evolves sublinearly for ADA. This difference leads to a substantial reduction in rounds for ADA, particularly for larger markets.

It is evident from Figure 2 that the number of rounds significantly depends on the value of c. To investigate this dependency further, we ran another set of experiments where we fixed $n = 1,000$ and varied c as $c = 0, 0.01, 0.02, \ldots, 1$. The results are reported in Figure 3 (for this experiment, we increased the number of markets per point to $10,000$ as the high variance was causing significant noise). As in Figure 2, the difference between the two algorithms is stark. For example, when $c = 0$, ADA reduces the number of rounds by a factor of 7. When $c = 0.5$, the scaling factor

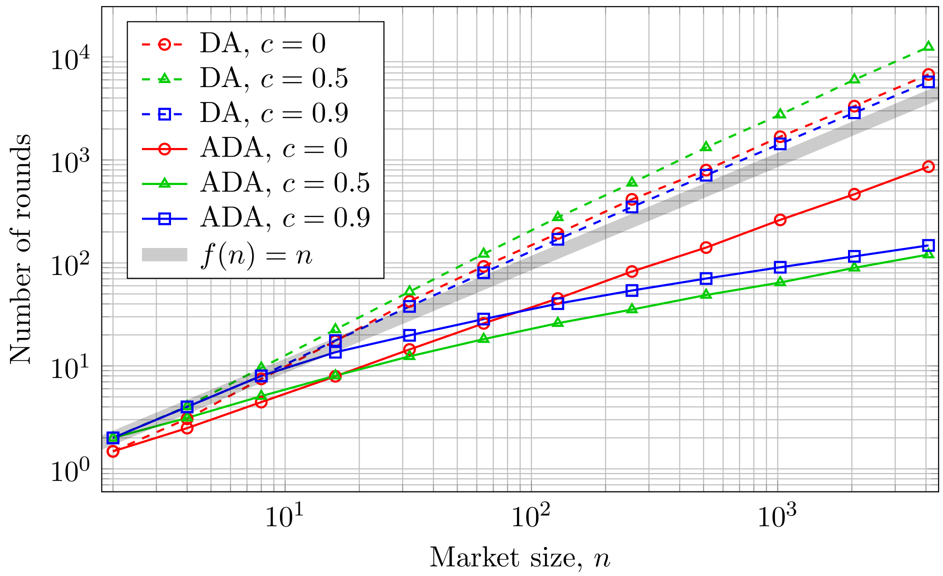

Figure 2: Number of rounds by n.

increases to 50. In fact, the values of c close to $c = 0.5$ seem to maximise the number of rounds for DA and minimise the number of rounds of ADA.

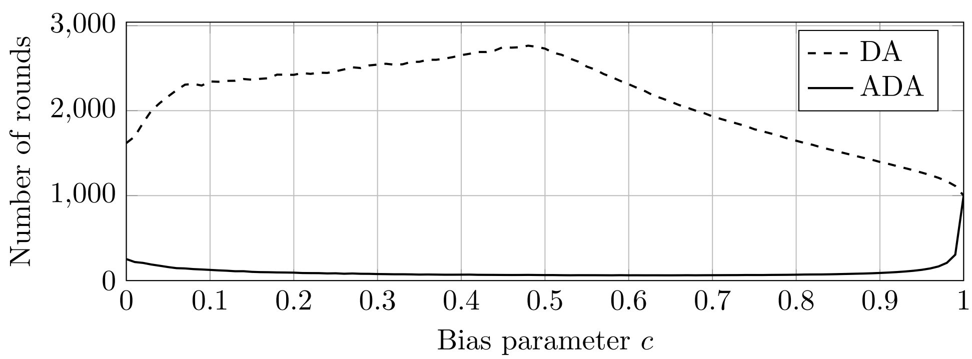

Figure 3: Number of rounds as a function of c.

As mentioned above, we find that the number of rounds needed by DA has significant variation. Figure 4 shows the distributions of the number of rounds for DA and ADA for uniformly random, moderately similar and similar markets. (Once again, we used 10,000 markets for each value of c for this experiment.)

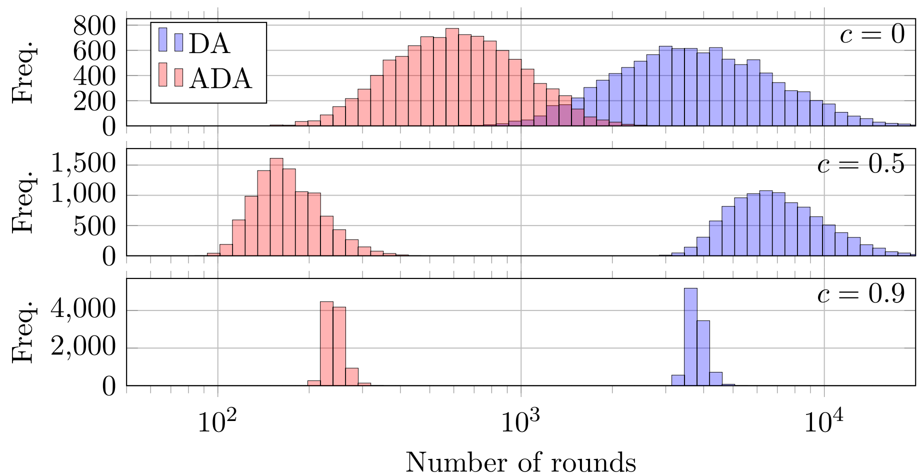

Figure 4: The distributions of the number of rounds for $c = 0.0$, $c = 0.5$ and $c = 0.9$.

An interesting observation from Figure 4 is that the two distributions for DA and ADA are similar except shifted. Since the scale of the horizontal axis is logarithmic, this means that the *relative* variances for the two algorithms are similar, however the *absolute* variances are very different. In other words, the number of rounds of ADA is far more predictable than the number of rounds of DA, which could be an important strength of ADA.

All the experiments so far demonstrated that ADA is superior to DA, on average, with respect to certain measures. However, it is natural to ask whether there exist individual markets that are "difficult" for one algorithm but not the other. To investigate this, we produced Figure 5 that shows how the number of rounds of DA is related to the number of rounds of ADA for each market ($1,000$ markets are used for each $c = 0, 0.5, 0.9$).

As is evident from Figure 5, the number of rounds required by the two algorithms are highly correlated. That is, markets where ADA "takes time" to clear also take time to clear using DA, and vice versa. Of course, we cannot rule out that there exists a collection of markets for which the performance of the two algorithms does not track. But we have not observed any in our experiments.

4.3 Number of proposals

Figures 6 and 7 show the number of proposals for each algorithm as it changes for n and c, respectively.

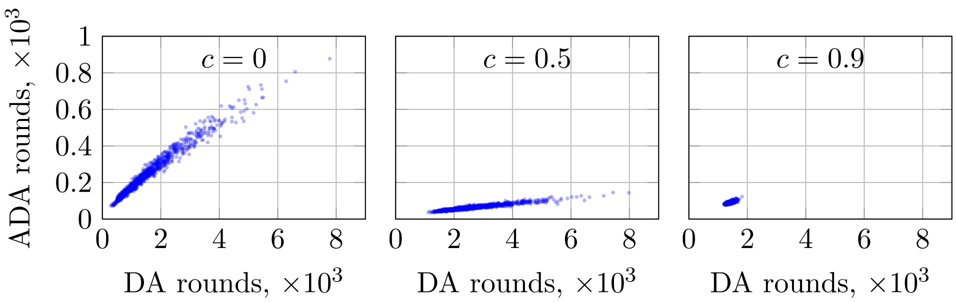

Figure 5: The relation between the number of rounds for DA and ADA.

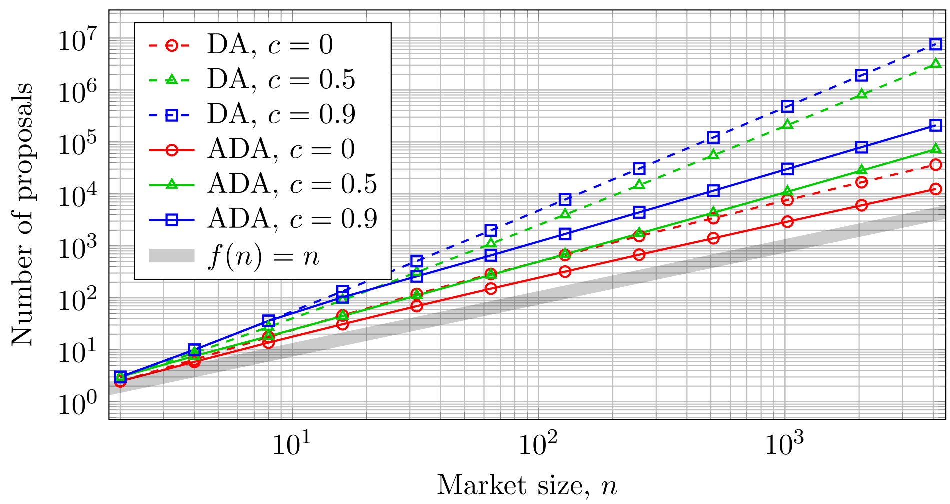

Figure 6: Number of proposals by n.

Unsurprisingly, the number of proposals required by both algorithms increases in n. However, for DA, the number of proposals needed dwarfs that of ADA. To see this, consider the largest market size, $n = 4,096$, with $c = 0.9$. On average DA used 7,637,702 proposals whereas ADA managed with only 208,585, a reduction of 96%.

Note that, unlike for number of rounds, the number of proposals monotonically increases with c. To understand the shape, it is useful to consider why they are exactly equal when $c = 1$. In this case, for both algorithms, all men propose to the same woman in round 1. She accepts her favourite. In round 2, all remaining $n - 1$ men propose to the agreed-upon second most desirable woman, who accepts her favourite. And so on. Continuing like

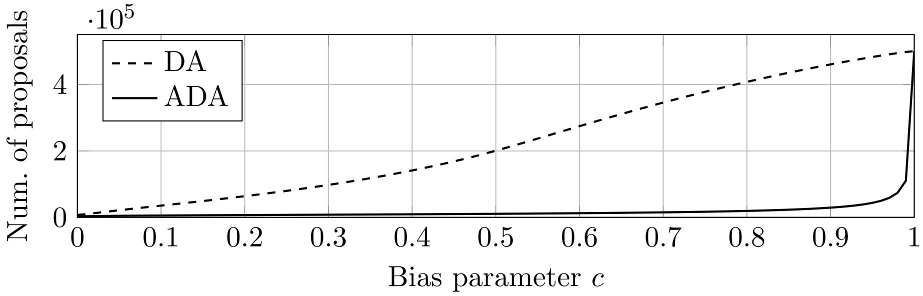

Figure 7: Number of proposals as a function of c.

this, it can be checked that both algorithms require $n(n+1)/2$ proposals to terminate. When c is close to but not equal to 1, DA proceeds in a manner similar to the above (but with slightly fewer concurrent proposals because some heterogeneity in preferences is present). However, even a small amount of heterogeneity in preferences can result in a lot of pre-emptive rejections for ADA which substantially reduces the number of proposals.

While not reported, we have checked the correlations for the numbers of proposals of DA and ADA, replicating the experiments at the end of Section 4.2. Our observations were similar except that the distributions for the number of proposals are in general narrower.

4.4 Proportion of final pairs matched by round

A metric that we believe is interesting is the "proportion of stable pairs matched by round". To see how the measure is computed, consider a market of size $n = 2$ where both men have identical preferences. One of the stable pairs meet in the first round, while the other pair do not meet until round 2. In this example, our metric describes the algorithm via a non-decreasing step function that takes the value 0.5 after the first round and the value 1 when all stable pairs have met. Figure 8 below plots this measure for both algorithms for $c = 0$, $c = 0.5$, and $c = 0.9$.

Consider the first panel in Figure 8 that corresponds to $c = 0$. Here, ADA arrives at the stable matching after 877 rounds while DA does not conclude until round $7,778$. After 877 rounds, DA has matched 90% of stable pairs. While close to completion on this measure, only 11% of rounds are finished. This can be contrasted with the third panel that plots the same measure for $c = 0.9$ (this same image was already presented in Figure 1 in Section 1). Here, upon termination of ADA, DA has completed 7% of rounds

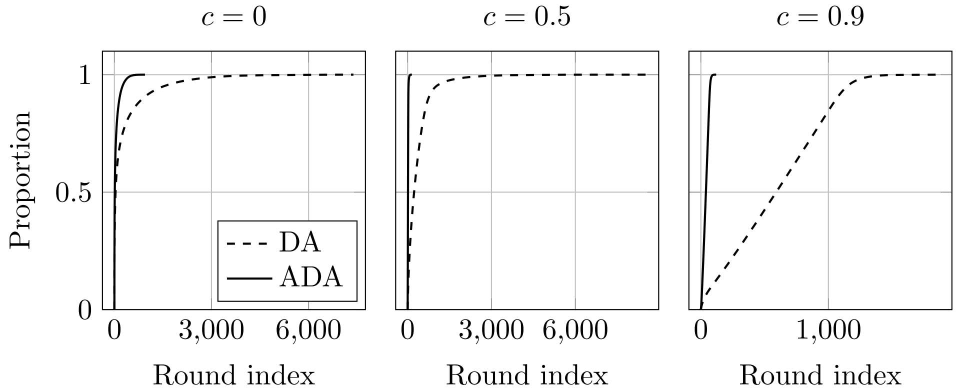

Figure 8: Proportion of final pairs matched by round.

but has only matched 13% of final pairs.

Despite the differences in rate of progression of the two algorithms across combinations of parameters, the shape of all six step functions plotted is similar. By definition all functions must be weakly increasing, but we note that all are concave. The concavity was not at all obvious to us in advance.

4.5 Execution time

Our final comparison is of the run times of both algorithms. Unfortunately, due to the lack of an industry-standard implementation of DA, we had to compare ADA to our own implementation of DA. Both were coded in Python and include only trivial optimisations. Indeed, since the difference between the two algorithms is relatively small, the implementations are also quite similar. For full transparency, and to enable future research, our implementations are publicly available at Karapetyan (2024).

Figure 9 plots average runtime as a function of c for $n = 1,000$. For both algorithms, the average runtime is strictly increasing in c, and the gap between their running times is also increasing with c except for $c > 0.9$. Other than the extreme point, $c = 1$, in our experiments ADA is always at least as fast as DA.

To investigate this speed up further, we plotted the distributions of the runtimes for the two algorithms in Figure 10. At least for our experiments, the runtime distributions are relatively narrow. In fact, for large c there is no overlap between them.

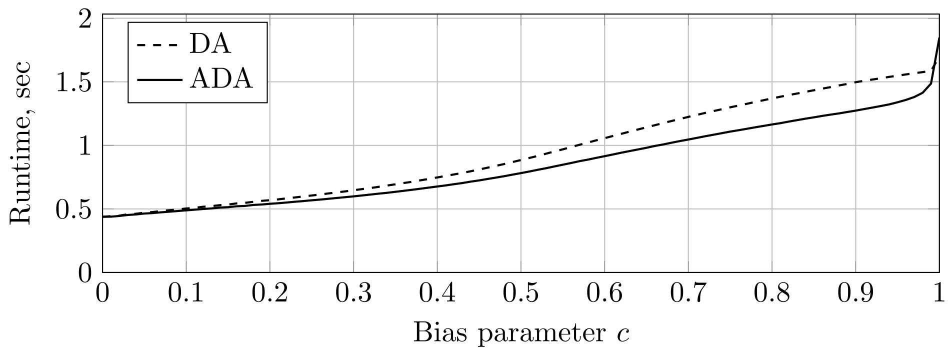

Figure 9: Runtime by c with $n = 1\,000$.

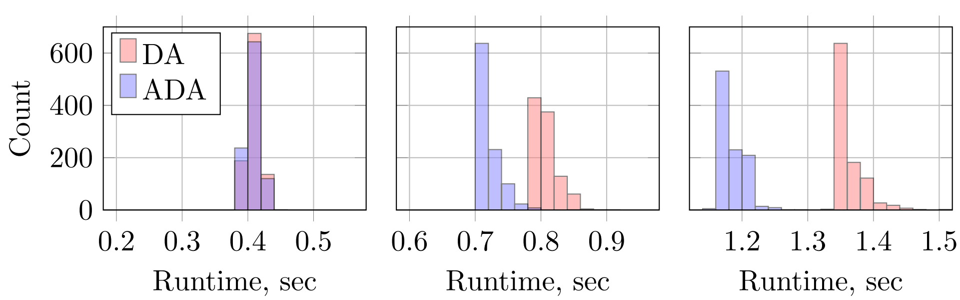

Figure 10: Runtime distributions for $n = 1,000$.

5 Conclusion and extensions for future work

Conclusion

The deferred acceptance (DA) algorithm of Gale and Shapley makes at least three outstanding contributions to the theory of matching. It confirms the existence of a stable matching for every two-sided market, it finds the optimal stable matching from the perspective of one side, and a mechanism based on it is *strategy proof* for the proposing side. The first property is fundamental. The second is of genuine practical importance because often, like when matching students to schools or doctors to hospitals, centralised matching protocols lean toward assignments that favour one side. The third means that achieving the second is not merely a pipe dream.

In this paper we proposed the accelerated deferred acceptance algorithm (ADA). This procedure differs from DA in one minor but important way: it rules out proposals that will knowingly never be accepted. Such an amendment does not change the output but it does generate efficiencies. ADA requires fewer proposals, terminates in fewer rounds, and stable pairings find each other earlier. In theory the efficiency gains are only weak. Computational experiments on simulated markets confirm that not only can these efficiency gains be strict but that they are often strict and can be substantial.

Extensions

We see two natural directions in which extensions are possible. The first involves investigating how ADA compares with DA in more general matching environments. The second entails including the "don't allow sure-to-be rejected proposals" insight to other variants of the DA procedure.

1. Evaluating ADA and DA in richer environments:

The matching environment that we have considered in this paper, Definition 1, is classified by the following three features.

(i) *a one-to-one market*: each participant can only match with one other.

(ii) *all preference lists are complete*: no participant views anyone on the other side of the market as unacceptable.

(iii) *a balanced market*: there are the same number of men as women.

The above features are far from universal to all matching markets. In fact, this environment is pretty artificial as all three seem unlikely to hold in practice. Our reasons for keeping these market characteristics fixed was twofold: (i) it seemed the natural starting point, and (ii) in order to keep the paper short (because there are so many ways that the above features can be relaxed).

Even amongst those that satisfy features (i) - (iii), there exist families of markets for which ADA terminates in a constant number of rounds, whereas DA requires an unbounded number. To give an example, consider a market where, for every $i = 1, \ldots, n-1$, woman w_i and man m_i rank each other first, but man m_n ranks woman w_n last. Here, ADA will wrap up after just two rounds while DA would require n rounds.

Disparities of this kind may be magnified when relaxing the features listed above, and so evaluating the performances of the two algorithms in richer environments seems a potentially interesting avenue of further study.

2. Accelerating other variants of deferred acceptance:

The key to our accelerated deferred acceptance algorithm is that it identifies proposals that will never be accepted because the proposee is currently matched to someone they prefer more. Mechanically, proposals of this nature are ruled out by truncating preference lists in response to top proposals, an insight that is borrowed from the IDUA procedure (Balinski and Ratier, 1997; Gutin et al., 2023). Our hope is that similar truncation operations might prove useful to existing variants of DA.

APPENDIX

A Proofs omitted from the main text

Theorem 2. Accelerated deferred acceptance never requires more proposals than DA.

Proof. Fix a two-sided matching market. Let p_D denote the number of proposals when running DA and let p_A denote the number of proposals when running ADA.

Let $\mu_M = \{(m_1, \tau(m_1)), \ldots, (m_n, \tau(m_n))\}$ be the man-optimal stable matching returned by both algorithms (recall, as per Theorem 1, that both algorithms return μ_M). By Observation 1, when DA terminates, each man m has proposed to all women that he strictly prefers to $\tau(m)$, and also to woman $\tau(m)$, and to no woman that he ranks below $\tau(m)$. Similarly, for ADA, no man m has proposed to any woman strictly worse than $\tau(m)$, since m would only propose to such a woman in the event that he was rejected by $\tau(m)$. However, under ADA each man m may not have proposed to all women strictly preferred to $\tau(m)$. Therefore, it must be that $p_A \leq p_D$. $\square$

Theorem 4. Each proposal that is made when running ADA takes place in the same round or an earlier round than when running DA.

The proof of Theorem 4 requires some terminology and notation. Say that (m, w) *is a pair* in matching problem P if w appears on the preference list of m and m appears on the preference list of w in P. Given a stable matching problem, P, we now define a sub-problem.

Definition 7. Say that matching problem P is a sub-problem of matching problem P' when the following properties hold.

1. if (m, w) is a pair in P, then it is also a pair in P'.
2. if man m prefers w over w' in P, then m also prefers w over w' in P'.
3. if woman w prefers m over m' in P, then w also prefers m over m' in P'.

We interpret both ADA and DA as deletion operations that generate sub-problems by deleting pairs. We are now ready to prove Theorem 4.

Proof. Let P be a stable matching problem. Let $A_k(P)$ denote the sub-problem obtained after k iterations of ADA and let $D_k(P)$ denote the sub-problem obtained after k iterations of DA. We will show by induction that $A_k(P)$ is a sub-problem $D_k(P)$ (denoted by $A_k(P) \subseteq D_k(P)$) for all $k = 0, 1, 2, \ldots$ By definition, $A_0(P) = D_0(P) = P$, and clearly $A_1(P) \subseteq D_1(P)$. Assume that $k \geq 2$, and, by the inductive hypothesis, assume that $A_r(P) \subseteq D_r(P)$ for all $r < k$.

Towards a contradiction, assume that $A_k(P)$ is not a sub-problem of $D_k(P)$. This implies that some pair (m, w) is a pair in $A_k(P)$ but not in $D_k(P)$. As $A_k(P) \subseteq A_{k-1}(P) \subseteq D_{k-1}(P)$, it must be that (m, w) is a pair in $D_{k-1}(P)$. This means that (m, w) is deleted from $D_{k-1}(P)$ by the DA algorithm in iteration k. Let (m', w) be a pair such that m' proposes to w in iteration k of DA but woman w prefers m' over m, which is the reason that (m, w) gets deleted from $D_{k-1}(P)$. We now consider the following two cases separately.

Case 1, (m', w) **is a pair in** $A_{k-1}(P)$**:** Since w is the most preferred woman for man m' in iteration k of DA, and $A_{k-1}(P) \subseteq D_{k-1}(P)$ (by induction), and (m', w) is a pair in $A_{k-1}(P)$ (by assumption), it must be that man m' also proposes to woman w in the k'th iteration of ADA. However, this implies that (m, w) is not a pair $A_k(P)$ because then woman w prefers m' over m, a contradiction to the assumption that (m, w) belongs to $A_k(P)$

Case 2, (m', w) **is not a pair in** $A_{k-1}(P)$**:** This implies that the pair (m', w) was deleted by ADA in the r'th iteration of ADA for some $r < k$. However, since woman w prefers m' over m this implies that (m, w) would also have been deleted in the r'th iteration (or earlier), a contradiction to the assumption that (m, w) belongs to $A_k(P)$.

In both of the above cases we obtain a contradiction, implying that $A_k(P) \subseteq D_k(P)$ for all non-negative integers k, as desired.

□

B Solving Example 1 using both algorithms

Example 1. Consider a one-to-one two-sided matching problem with five men, $M = \{m_1, m_2, m_3, m_4, m_5\}$, and five women, $W = \{w_1, w_2, w_3, w_4, w_5\}$. The preference lists for each man and each woman are presented below, with potential partners listed in order of decreasing preference.

$w_1 : m_5, m_4, m_1, m_2, m_3$ $\qquad$ $m_1 : w_1, w_2, w_3, w_4, w_5$

$w_2 : m_1, m_3, m_2, m_4, m_5$ $\qquad$ $m_2 : w_1, w_4, w_5, w_2, w_3$

$w_3 : m_5, m_4, m_3, m_2, m_1$ $\qquad$ $m_3 : w_1, w_4, w_3, w_5, w_2$

$w_4 : m_4, m_2, m_1, m_3, m_5$ $\qquad$ $m_4 : w_4, w_2, w_3, w_1, w_5$

$w_5 : m_5, m_1, m_3, m_4, m_2$ $\qquad$ $m_5 : w_5, w_4, w_1, w_2, w_3$

Round 1:

w_1		w_2		w_3		w_4		w_5	
m_1	m_1					m_4	m_4	m_5	m_5
m_2^*	m_2^*								
m_3^*	m_3^*								

$w_1 : m_5, m_4, m_1, \boxed{m_2}, \boxed{m_3}$ $\qquad$ $m_1 : w_1, w_2, w_3, w_4, w_5$

$w_2 : m_1, m_3, m_2, m_4, m_5$ $\qquad$ $m_2 : \boxed{w_1}, w_4, w_5, w_2, w_3$

$w_3 : m_5, m_4, m_3, m_2, m_1$ $\qquad$ $m_3 : \boxed{w_1}, w_4, w_3, w_5, w_2$

$w_4 : m_4, m_2, m_1, m_3, m_5$ $\qquad$ $m_4 : w_4, w_2, w_3, w_1, w_5$

$w_5 : m_5, m_1, m_3, m_4, m_2$ $\qquad$ $m_5 : w_5, w_4, w_1, w_2, w_3$

Round 2:

w_1		w_2		w_3		w_4		w_5	
m_1	m_1		m_2		m_3	m_4	m_4	m_5	m_5
						m_3^*			
						m_2^*			

$w_1 : m_5, m_4, m_1, \boxed{m_2}, \boxed{m_3}$ $\qquad$ $m_1 : w_1, w_2, w_3, w_4, w_5$

$w_2 : m_1, m_3, m_2, m_4, m_5$ $\qquad$ $m_2 : \boxed{w_1}, \boxed{w_4}, w_5, w_2, w_3$

$w_3 : m_5, m_4, m_3, m_2, m_1$ $\qquad$ $m_3 : \boxed{w_1}, \boxed{w_4}, w_3, w_5, w_2$

$w_4 : m_4, \boxed{m_2}, m_1, \boxed{m_3}, m_5$ $\qquad$ $m_4 : w_4, w_2, w_3, w_1, w_5$

$w_5 : m_5, m_1, m_3, m_4, m_2$ $\qquad$ $m_5 : w_5, w_4, w_1, w_2, w_3$

Since ADA has terminated we focus only on DA from here on. The single men at the beginning of Round 3 are m_2 and m_3. Man m_2's top choice is w_5 (note that w_5 has already rejected m_2 in ADA indicated by writing w_5 instead of w_5 in m_2's preference list) and next on m_3's list is the as yet unproposed to w_3. So, in Round 3 of DA, m_2 proposes to and is rejected by w_5 (as forecast by ADA) and so will be back on the market in Round 4 and m_3 proposes to w_3 who tentatively accepts him given that she is single.

Round 3:

w_1		w_2		w_3		w_4		w_5	
m_1	m_1		m_2	m_3	m_3	m_4	m_4	m_5	m_5
								m_2^*	

$$\begin{array}{ll} w_1: m_5, m_4, m_1, \boxed{m_2}, \boxed{m_3} & m_1: w_1, w_2, w_3, w_4, w_5 \\ w_2: m_1, m_3, m_2, m_4, m_5 & m_2: \boxed{w_1}, \boxed{w_4}, \boxed{w_5}, w_2, w_3 \\ w_3: m_5, m_4, m_3, m_2, m_1 & m_3: \boxed{w_1}, \boxed{w_4}, w_3, w_5, w_2 \\ w_4: m_4, \boxed{m_2}, m_1, \boxed{m_3}, m_5 & m_4: w_4, w_2, w_3, w_1, w_5 \\ w_5: m_5, m_1, m_3, m_4, \boxed{m_2} & m_5: w_5, w_4, w_1, w_2, w_3 \end{array} \tag{5}$$

The only man without a partner at the beginning of round 4 of DA is m_2, who was rejected by woman w_5 in round 3. This rejection is indicated by the red box around m_2 in the preference list of w_5 and the red box around w_5 in the preference list of m_2. The top ranked woman on m_2's preference list that he has not thus far proposed to is w_2. Woman w_2 is currently unmatched and so she (tentatively) accepts m_2. Every woman now has a tentative match and so the algorithm terminates and returns the collection of pairs that make up a matching.

Round 4:

w_1		w_2		w_3		w_4		w_5	
m_1	m_1	m_2	m_2	m_3	m_3	m_4	m_4	m_5	m_5

References

Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, June 2003.

Mohammad Akbarpour, Shengwu Li, and Shayan Oveis Gharan. Thickness and information in dynamic matching markets. *Journal of Political Economy*, 128(3):783–815, 2020.

Michel Balinski and Guillaume Ratier. Of stable marriages and graphs, and strategy and polytopes. *SIAM Review*, 39(4):575–604, 1997.

Inácio Bó and Rustamdjan Hakimov. Iterative Versus Standard Deferred Acceptance: Experimental Evidence. *The Economic Journal*, 130(626): 356–392, 07 2019.

Inácio Bó and Rustamdjan Hakimov. The iterative deferred acceptance mechanism. *Games and Economic Behavior*, 135:411–433, 2022.

Ken Burdett and Melvyn G. Coles. Marriage and class. *The Quarterly Journal of Economics*, 112(1):141–168, 02 1997.

Claudia Cerrone, Yoan Hermstrüwer, and Onur Kesten. School Choice with Consent: an Experiment. *The Economic Journal*, 134(661):1760–1805, 01 2024.

Harold L. Cole, George J. Mailath, and Andrew Postlewaite. Social norms, savings behavior, and growth. *Journal of Political Economy*, 100(6):1092–1125, 1992.

Vincent P. Crawford and Elsie Marie Knoer. Job matching with heterogeneous firms and workers. *Econometrica*, 49(2):437–450, 1981.

Vânia M.F. Dias, Guilherme D. da Fonseca, Celina M.H. de Figueiredo, and Jayme L. Szwarcfiter. The stable marriage problem with restricted pairs. *Theoretical Computer Science*, 306(1):391–405, 2003.

Laura Doval. Dynamically stable matching. *Theoretical Economics*, 17(2): 687–724, 2022.

L. E. Dubins and D. A. Freedman. Machiavelli and the gale-shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494, 1981.

Federico Echenique, Nicole Immorlica, and Vijay V. Vazirani. *Online and Matching-Based Market Design*. Cambridge University Press, 2023.

Jan Eeckhout. Bilateral search and vertical heterogeneity. *International Economic Review*, 40(4):869–887, 1999.

D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

David Gale and Marilda Sotomayor. Ms. machiavelli and the stable matching problem. *The American Mathematical Monthly*, 92(4):261–268, 1985a.

David Gale and Marilda Sotomayor. Some remarks on the stable matching problem. *Discrete Applied Mathematics*, 11(3):223–232, 1985b.

Kartik Gokhale, Amit Kumar Mallik, Ankit Kumar Misra, and Swaprava Nath. A gale-shapley view of unique stable marriages. In *ECAI 2024*, pages 3652–3659. IOS Press, 2024.

Yannai A. Gonczarowski, Ori Heffetz, and Clayton Thomas. Strategyproofness-exposing mechanism descriptions, 2023. URL https://arxiv.org/abs/2209.13148.

Yannai A. Gonczarowski, Ori Heffetz, Guy Ishai, and Clayton Thomas. Describing deferred acceptance and strategyproofness to participants: Experimental analysis, 2024. URL https://arxiv.org/abs/2409.18166.

Dan Gusfield and Robert W. Irving. *The Stable Marriage Problem: Structure and Algorithms.* MIT Press, Cambridge, MA, USA, 1989. ISBN 0262071185.

Gregory Z. Gutin, Philip R. Neary, and Anders Yeo. Unique stable matchings. *Games and Economic Behavior*, 141:529–547, 2023.

Gregory Z. Gutin, Philip R. Neary, and Anders Yeo. Finding all stable matchings with assignment constraints. *Games and Economic Behavior*, 148:244–263, 2024.

Peter J Hammond. Straightforward individual incentive compatibility in large economies. *The Review of Economic Studies*, 46(2):263–282, 1979.

F. A. Hayek. The use of knowledge in society. *American Economic Review*, 1945.

Ron Holzman and Dov Samet. Matching of like rank and the size of the core in the marriage problem. *Games and Economic Behavior*, 88:277–285, 2014.

Daniel Karapetyan. Source code of the stable matching problem algorithms and market generator. http://doi.org/10.17639/nott.7463, 2024.

Jr Kelso, Alexander S and Vincent P Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, November 1982.

Onur Kesten. School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348, 08 2010.

D.E. Knuth. *Stable Marriage and Its Relation to Other Combinatorial Problems: An Introduction to the Mathematical Analysis of Algorithms*. CRM proceedings & lecture notes. American Mathematical Soc., 1996.

D. G. McVitie and L. B. Wilson. Stable marriage assignment for unequal sets. *BIT Numerical Mathematics*, 10(3):295–309, 1970.

D. G. McVitie and L. B. Wilson. The stable marriage problem. *Commun. ACM*, 14(7):486–490, July 1971.

Timo Mennle and Sven Seuken. Trade-offs in school choice: Comparing deferred acceptance, the naive and the classic boston mechanism, 2017. URL https://arxiv.org/abs/1406.3327.

Paul Milgrom and Ilya Segal. Clock auctions and radio spectrum reallocation. *Journal of Political Economy*, 128(1):1–31, 2020.

Rosemarie Nagel. Unraveling in guessing games: An experimental study. *American Economic Review*, 85(5):1313–26, December 1995.

Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, New York, NY, USA, 2007.

Josue Ortega, Gabriel Ziegler, and R. Pablo Arribillaga. Unimprovable students and inequality in school choice, 2024. URL https://arxiv.org/abs/2407.19831.

Alvin E. Roth. The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7(4):617–628, 1982.

Alvin E. Roth. The evolution of the labor market for medical interns and residents: A case study in game theory. *Journal of Political Economy*, 92 (6):991–1016, 1984.

Alvin E. Roth. On the allocation of residents to rural hospitals: A general property of two-sided matching markets. *Econometrica*, 54(2):425–427, 1986.

Alvin E. Roth. Deferred acceptance algorithms: history, theory, practice, and open questions. *International Journal of Game Theory*, 36(3):537–569, 2008.

Dale O. Stahl and Paul W. Wilson. Experimental evidence on players' models of other players. *Journal of Economic Behavior and Organization*, 25(3):309 – 327, 1994.

Dale O. Stahl and Paul W. Wilson. On players' models of other players: Theory and experimental evidence. *Games and Economic Behavior*, 10 (1):218 – 254, 1995.

Qianfeng Tang and Jingsheng Yu. A new perspective on kesten's school choice with consent idea. *Journal of Economic Theory*, 154:543–561, 2014.